Set Theory as an Axiomatic System
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Constructing the Notation and Axioms

To work with sets formally, within the predicate calculus, we can adopt a technical vocabulary
that includes the binary predicates \(\text{Element-of}\), \(\text{Subset-of}\), and \(\text{Strict-subset-of}\), binary functions
\(\text{union}\), \(\text{intersection}\), and \(\text{difference}\), unary function \(\wp\) (for “power set”), and a term \(\emptyset\). We’ll also
subsume the theory of identity into our set theory, with its binary predicate \(I\) and its axioms and
rules of inference, including the axioms and rules of inference of the predicate calculus.

As is customary, and to bring our notation in line with the textbook, we’ll write
\[
\lfloor (\sigma \in \tau) \rfloor
\]
for \(\lfloor \text{Element-of}(\sigma, \tau) \rfloor\) (for any terms \(\lfloor \sigma \rfloor\) and \(\lfloor \tau \rfloor\)), and similarly
\[
\lfloor (\sigma \subseteq \tau) \rfloor
\]
for \(\lfloor \text{Subset-of}(\sigma, \tau) \rfloor\) and \(\lfloor (\sigma \subset \tau) \rfloor\) for \(\lfloor \text{Strict-subset-of}(\sigma, \tau) \rfloor\). Likewise for the binary functions:
\[
\lfloor (\sigma \cup \tau) \rfloor
\]
is a shorthand form for \(\lfloor \text{union}(\sigma, \tau) \rfloor\), \(\lfloor (\sigma \cap \tau) \rfloor\) for \(\lfloor \text{intersection}(\sigma, \tau) \rfloor\), and \(\lfloor (\sigma - \tau) \rfloor\) for \(\lfloor \text{difference}(\sigma, \tau) \rfloor\).

Next, we need axioms and inference rules that define all of these new symbols. All but one of
these are really just formalizations of the informal definitions from section 4.1 of the textbook.

- \(\vdash ((\sigma \subseteq \tau) \iff \forall \xi (\xi \in \sigma \rightarrow \xi \in \tau))\) (definition of subset).
- \(\vdash ((\sigma \subset \tau) \iff ((\sigma \subseteq \tau) \land \neg(\sigma = \tau)))\) (definition of strict subset).
- \(\vdash \forall \xi (((\xi \in (\sigma \cup \tau)) \iff ((\xi \in \sigma) \lor (\xi \in \tau)))\) (definition of union).
- \(\vdash \forall \xi (((\xi \in (\sigma \cap \tau)) \iff ((\xi \in \sigma) \land (\xi \in \tau)))\) (definition of intersection).
- \(\vdash \forall \xi (((\xi \in (\sigma - \tau)) \iff ((\xi \in \sigma) \land \neg(\xi \in \tau)))\) (definition of set difference).
- \(\vdash \forall \xi (\neg(\xi \in \emptyset))\) (definition of empty set).
- \(\vdash ((\sigma = \tau) \iff \forall \xi ((\xi \in \sigma) \iff (\xi \in \tau)))\) (axiom of set equality).

The last of these axioms is an observation, intended to be obviously and incontestably true,
about what constitutes the identity of a set, namely, its membership. If \(\sigma\) and \(\tau\) have exactly
the same elements, then \(\sigma = \tau\).

The axiom of set equality in this form means that the theory will only be true when the
universe of discourse consists entirely of sets. For if there were any non-set entities in the universe
of discourse, they would presumably have no elements. Under the axiom of set equality, then, they
would all be identical to \(\emptyset\) and so would be sets after all.

There is a kind of workaround for this, suggested by the logician Willard van Orman Quine in
his book *Set Theory and Its Logic*, where he calls non-set entities “individuals.” For any individual
\(x\), we can adopt the convention that \((x = \{x\})\): An individual is the set that has that very individual
as its only element. It turns out that this convention is consistent with the other axioms so long
as we don’t mistakenly try to apply it to ordinary sets such as \(\emptyset\), which cannot be individuals in
Quine’s sense. (The empty set, by definition, has no elements at all; it is a subset of itself, but not
an element of itself.)
Proofs in Axiomatic Set Theory

Theorem 1. To prove: \( \forall x \forall y (x \subseteq (x \cup y)) \).

Lemma A: \((c \in a) \vdash (c \in (a \cup b))\).

(0) \((c \in a)\) premiss
(1) \((c \in a) \lor (c \in b)\) (0), disjunction introduction
(2) \(\forall x ((x \in (a \cup b)) \leftrightarrow ((x \in a) \lor (x \in b)))\) definition of union
(3) \((c \in (a \cup b)) \leftrightarrow ((c \in a) \lor (c \in b))\) (2), universal quantifier elimination
(4) \((c \in (a \cup b))\) (1), (3), equivalence elimination (right-to-left)

Lemma B: \(\vdash ((c \in a) \rightarrow (c \in (a \cup b)))\).

(0) \(((c \in a) \rightarrow (c \in (a \cup b)))\) lemma A, implication introduction

Lemma C: \(\vdash \forall y (a \subseteq (a \cup y))\).

(0) \(\forall y (a \subseteq (a \cup y))\) lemma C, universal quantifier introduction

Main proof of proposition:

(0) \(\forall y (a \subseteq (a \cup y))\) lemma D, universal quantifier introduction

Theorem 2. To prove: \(\forall x ((\emptyset - x) = \emptyset)\).

Lemma A: \((b \in (\emptyset - a)) \vdash (b \in \emptyset)\).

(0) \((b \in (\emptyset - a))\) premiss
(1) \((b \in (\emptyset - a)) \leftrightarrow ((b \in \emptyset) \land \neg(x \in a))\) definition of set difference
(2) \((b \in (\emptyset - a)) \leftrightarrow ((b \in \emptyset) \land \neg(b \in a))\) (1), universal quantifier elimination
(3) \((b \in \emptyset \land \neg b \in a)\) (0), (2), modus ponens
(4) \((b \in \emptyset)\) (3), conjunction elimination, left

Lemma B: \((b \in \emptyset), \neg(b \in (\emptyset - a)) \vdash \text{F}\).

(0) \((b \in \emptyset)\) premiss
(1) \(\neg(b \in (\emptyset - a))\) premiss
(2) \(\forall x \neg(x \in \emptyset)\) definition of empty set
(3) \(\neg(b \in \emptyset)\) (2), universal quantifier elimination
(4) \(\text{F}\) (0), (3), catastrophe of contradiction

Lemma C: \((b \in \emptyset) \vdash (b \in (\emptyset - a))\).

(0) \((b \in \emptyset)\) premiss
(1) \(\neg(b \in (\emptyset - a))\) (0), lemma B, reductio ad absurdum
(2) \((b \in (\emptyset - a))\) (1), negation elimination
Lemma D: \( \vdash ((b \in (\emptyset - a)) \leftrightarrow (b \in \emptyset)) \).

\[
(0) \quad ((b \in (\emptyset - a)) \leftrightarrow (b \in \emptyset)) \quad \text{lemmas A and C, equivalence introduction}
\]

Lemma E: \( ((\emptyset - a) = \emptyset) \).

\[
(0) \quad \forall x ((x \in (\emptyset - a)) \leftrightarrow (x \in \emptyset)) \quad \text{lemma D, universal quantifier introduction}
\]

\[
(1) \quad (((\emptyset - a) = \emptyset) \leftrightarrow \forall x ((x \in (\emptyset - a)) \leftrightarrow (x \in \emptyset))) \quad \text{axiom of set equality}
\]

\[
(2) \quad ((\emptyset - a) = \emptyset) \quad (0), (1), \text{equivalence elimination (right-to-left)}
\]

Main proof of the proposition:

\[
(0) \quad \forall x ((\emptyset - x) = \emptyset) \quad \text{lemma E, universal quantifier introduction}
\]

**Theorem 3.** To prove: \( \vdash \forall x \forall y \forall z ((x \cap (y \cup z)) = ((x \cap y) \cup (x \cap z))) \).

This is a distributive law for sets: Union distributes over intersection. In the textbook, it is proven as Theorem 4.1.2.

Lemma A: \( (d \in a), (d \in b) \vdash ((d \in (a \cap b)) \vee (d \in (a \cap c))) \).

\[
(0) \quad (d \in a) \quad \text{premise}
\]

\[
(1) \quad (d \in b) \quad \text{premise}
\]

\[
(2) \quad ((d \in a) \land (d \in b)) \quad (0), (1), \text{conjunction introduction}
\]

\[
(3) \quad \forall x ((x \in (a \cap b)) \leftrightarrow ((x \in a) \land (x \in b))) \quad \text{definition of intersection}
\]

\[
(4) \quad ((d \in (a \cap b)) \leftrightarrow ((d \in a) \land (d \in b))) \quad (3), \text{universal quantifier elimination}
\]

\[
(5) \quad (d \in (a \cap b)) \quad (2), (4), \text{equivalence elimination (right-to-left)}
\]

\[
(6) \quad ((d \in (a \cap b)) \vee (d \in (a \cap c))) \quad (5), \text{disjunction introduction, left}
\]

Lemma B: \( (d \in a), (d \in c) \vdash ((d \in (a \cap b)) \vee (d \in (a \cap c))) \).

\[
(0) \quad (d \in a) \quad \text{premise}
\]

\[
(1) \quad (d \in c) \quad \text{premise}
\]

\[
(2) \quad ((d \in a) \land (d \in c)) \quad (0), (1), \text{conjunction introduction}
\]

\[
(3) \quad \forall x ((x \in (a \cap c)) \leftrightarrow ((x \in a) \land (x \in c))) \quad \text{definition of intersection}
\]

\[
(4) \quad ((d \in (a \cap c)) \leftrightarrow ((d \in a) \land (d \in c))) \quad (3), \text{universal quantifier elimination}
\]

\[
(5) \quad (d \in (a \cap c)) \quad (2), (4), \text{equivalence elimination (right-to-left)}
\]

\[
(6) \quad ((d \in (a \cap b)) \vee (d \in (a \cap c))) \quad (5), \text{disjunction introduction, right}
\]

Lemma C: \( (d \in (a \cap (b \cup c))) \vdash (d \in ((a \cap b) \cup (a \cap c))) \).

\[
(0) \quad (d \in (a \cap (b \cup c))) \quad \text{premise}
\]

\[
(1) \quad \forall x ((x \in (a \cap (b \cup c))) \leftrightarrow ((x \in a) \land (x \in (b \cup c)))) \quad \text{definition of intersection}
\]

\[
(2) \quad ((d \in (a \cap (b \cup c))) \leftrightarrow ((d \in a) \land (d \in (b \cup c)))) \quad (1), \text{universal quantifier elimination}
\]

\[
(3) \quad ((d \in a) \land (d \in (b \cup c))) \quad (0), (2), \text{equivalence elimination (left-to-right)}
\]

\[
(4) \quad (d \in a) \quad (3), \text{conjunction elimination (left)}
\]

\[
(5) \quad (d \in (b \cup c)) \quad (3) \text{conjunction elimination (right)}
\]

\[
(6) \quad ((d \in (a \cap b)) \vee (d \in (a \cap c))) \quad (4), (5), \text{lemmas A and B, disjunctive syllogism}
\]

\[
(7) \quad \forall x ((x \in ((a \cap b) \cup (a \cap c))) \leftrightarrow ((x \in (a \cap b)) \vee (x \in (a \cap c)))) \quad \text{definition of union}
\]

\[
(8) \quad ((d \in ((a \cap b) \cup (a \cap c))) \leftrightarrow ((d \in (a \cap b)) \vee (d \in (a \cap c)))) \quad (7), \text{universal quantification elimination}
\]

\[
(9) \quad ((d \in ((a \cap b) \cup (a \cap c))) \leftrightarrow ((d \in (a \cap b)) \vee (d \in (a \cap c)))) \quad (6), (8), \text{equivalence elimination (right-to-left)}
\]
Lemma D: \((d \in (a \cap b)) \vdash (d \in a) \land (d \in (b \cup c)))\).

(0) \((d \in (a \cap b))\) \hspace{1cm} \text{premiss}
(1) \ \forall x ((x \in (a \cap b)) \iff ((x \in a) \land (x \in b))) \hspace{1cm} \text{definition of intersection}
(2) \ ((d \in (a \cap b)) \iff ((d \in a) \land (d \in b))) \hspace{1cm} (1), \text{universal quantifier elimination}
(3) \ ((d \in a) \land (d \in b)) \hspace{1cm} (0), (2), \text{equivalence elimination (left-to-right)}
(4) \ (d \in a) \hspace{1cm} (3), \text{conjunction elimination (left)}
(5) \ (d \in b) \hspace{1cm} (3), \text{conjunction elimination (right)}
(6) \ ((d \in b) \lor (d \in c)) \hspace{1cm} (5), \text{disjunction introduction, left}
(7) \ \forall x ((x \in (b \cup c)) \iff ((x \in b) \lor (x \in c))) \hspace{1cm} \text{definition of union}
(8) \ ((d \in (b \cup c)) \iff ((d \in b) \lor (d \in c))) \hspace{1cm} (7), \text{universal quantifier elimination}
(9) \ (d \in (b \cup c)) \hspace{1cm} (6), (8), \text{equivalence elimination (right-to-left)}

Lemma E: \((d \in (a \cap c)) \vdash (d \in a) \land (d \in (b \cup c)))\).

(0) \((d \in (a \cap c))\) \hspace{1cm} \text{premiss}
(1) \ \forall x ((x \in (a \cap c)) \iff ((x \in a) \land (x \in c))) \hspace{1cm} \text{definition of intersection}
(2) \ ((d \in (a \cap c)) \iff ((d \in a) \land (d \in c))) \hspace{1cm} (1), \text{universal quantifier elimination}
(3) \ ((d \in a) \land (d \in c)) \hspace{1cm} (0), (2), \text{equivalence elimination (left-to-right)}
(4) \ (d \in a) \hspace{1cm} (3), \text{conjunction elimination (left)}
(5) \ (d \in c) \hspace{1cm} (3), \text{conjunction elimination (right)}
(6) \ ((d \in b) \lor (d \in c)) \hspace{1cm} (5), \text{disjunction introduction, right}
(7) \ \forall x ((x \in (b \cup c)) \iff ((x \in b) \lor (x \in c))) \hspace{1cm} \text{definition of union}
(8) \ ((d \in (b \cup c)) \iff ((d \in b) \lor (d \in c))) \hspace{1cm} (7), \text{universal quantifier elimination}
(9) \ (d \in (b \cup c)) \hspace{1cm} (6), (8), \text{equivalence elimination (right-to-left)}

Lemma F: \((d \in ((a \cap b) \cup (a \cap c))) \vdash (d \in (a \cap (b \cup c)))\).

(0) \((d \in ((a \cap b) \cup (a \cap c)))\) \hspace{1cm} \text{premiss}
(1) \ \forall x ((x \in ((a \cap b) \cup (a \cap c))) \iff ((x \in (a \cap b)) \lor (x \in (a \cap c))) \hspace{1cm} \text{definition of union}
(2) \ ((d \in ((a \cap b) \cup (a \cap c))) \iff ((d \in (a \cap b)) \lor (d \in (a \cap c)))) \hspace{1cm} (1), \text{universal quantifier elimination}
(3) \ ((d \in (a \cap b)) \lor (d \in (a \cap c))) \hspace{1cm} (0), (2), \text{equivalence elimination (right-to-left)}
(4) \ ((d \in a) \land (d \in (b \cup c))) \hspace{1cm} (3), \text{lemmas D and E, disjunctive syllogism}
(5) \ \forall x ((x \in (a \cap (b \cup c))) \iff ((x \in a) \land (x \in (b \cup c))) \hspace{1cm} \text{definition of intersection}
(6) \ ((d \in (a \cap (b \cup c))) \iff ((d \in a) \land (d \in (b \cup c))) \hspace{1cm} (5), \text{universal quantifier elimination}
(7) \ (d \in (a \cap (b \cup c))) \hspace{1cm} (4), (6), \text{equivalence elimination (right-to-left)}

Lemma G: \(\vdash ((d \in (a \cap (b \cup c))) \iff (d \in ((a \cap b) \cup (a \cap c))))\).

(0) \((d \in ((a \cap b) \cup (a \cap c)))\) \hspace{1cm} \text{lemmas C and F, equivalence introduction}

Lemma H: \(\vdash ((a \cap (b \cup c)) = ((a \cap b) \cup (a \cap c)))\).

(0) \ \forall x ((x \in (a \cap (b \cup c))) \iff (x \in ((a \cap b) \cup (a \cap c))) \hspace{1cm} \text{lemma G, universal quantified introduction}
(1) \(((a \cap (b \cup c)) = ((a \cap b) \cup (a \cap c))) \iff \forall x ((x \in (a \cap (b \cup c))) \iff (x \in ((a \cap b) \cup (a \cap c)))) \hspace{1cm} \text{axiom of set equality}
(2) \((a \cap (b \cup c)) = ((a \cap b) \cup (a \cap c))) \hspace{1cm} (0), (1), \text{equivalence elimination (right-to-left)}
Lemma I: \( \vdash \forall z ((a \cap (b \cup z)) = ((a \cap b) \cup (a \cap z))). \)

(0) \( \forall z ((a \cap (b \cup z)) = ((a \cap b) \cup (a \cap z))) \quad \text{lemma } H, \text{ universal quantifier introduction} \)

Lemma J: \( \vdash \forall y \forall z ((a \cap (y \cup z)) = ((a \cap y) \cup (a \cap z))). \)

(0) \( \forall y \forall z ((a \cap (y \cup z)) = ((a \cap y) \cup (a \cap z))) \quad \text{lemma } I, \text{ universal quantifier introduction} \)

Main proof of proposition:

(0) \( \forall x \forall y \forall z ((x \cap (y \cup z)) = ((x \cap y) \cup (x \cap z))) \quad \text{lemma } J, \text{ universal quantifier introduction} \)