As we have seen, the Product Rule guides us to the answer to the question of how many different sequences of length \( n \) we can construct from a set of \( n \) elements without using the same value twice. We can choose any of the \( n \) values to be the first element. For the second element, we can choose any of the remaining \( n - 1 \) values. And so on through the sequence: At the \( k \)th step, we can choose any of the \( n - k + 1 \) values that haven’t yet been used. For the last element of the sequence, we have only one unused value, so there is only one option in that case. Each sequence of decisions leads to a different arrangement of the \( n \) values, so the Product Rule tells us that there are \( n \times (n - 1) \times \ldots \times 1 \) such arrangements.

This particular product—the product of the positive integers up to and including \( n \)—comes up so often in counting problems that there is a name and a notation for it: It is called the factorial of \( n \), written ‘\( n! \)’. There is no simple closed form that can be used to compute the exact value of \( n! \) without doing all of the multiplications, but the mathematician James Stirling found an approximation that is sometimes useful:

\[
  n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
\]

As \( n \) increases without limit, the ratio of the two sides of this approximation approaches 1 as a limit:

\[
  \lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1
\]

Returning to the mechanics of sequence construction, we could ask a more general question by allowing the length of the sequence to vary: If we’re again given a set of \( n \) different values, how many sequences of length \( k \) can we form without repeating any value? If \( k > n \), the answer is of course “none” (there would be more positions to fill than values to fill them with), and we’ve seen that when \( k = n \) the answer is \( n! \). But what if \( k < n \)? In this case, the Product Rule gives the answer \( n \times (n - 1) \times \ldots \times (n - k + 1) \), which is called the \( k \)th falling factorial power of \( n \), written ‘\( n^\wedge{k} \).

There’s an obvious relation between factorials and falling factorial powers:

\[
  n^\wedge{k} = \frac{n!}{(n - k)!}
\]

provided that \( k \leq n \). The idea is that dividing by \( (n - k)! \) cancels all the unwanted small multipliers in \( n! \), leaving only the big ones from \( n \) down to \( n - k + 1 \).

Finally, let’s consider what happens when we select \( k \) values from our collection of \( n \) different values, without placing them in a sequence, but just designating them in some way—appointing a committee of \( k \) members within an organization that has \( n \) members altogether, say. How may different ways are there of constructing such a committee?

Well, we just found that if we choose the \( k \) members one by one, removing each one from the class of available options at the end of each step, we get \( n^\wedge{k} \) possible results. Since this time we’re considering each of these results just as a selection and not as a sequence, however, there will be duplicates. In fact, we generate each of the \( k \)-member committees in every possible order, and we know from our analysis of the problem of arrangements above that there will be \( k! \) such orders.
Thus our $n^k$ results include $k!$ copies of each $k$-member committee. By the Division Rule, therefore, the number of distinct committees is $\frac{n^k}{k!}$.

This quantity is called the binomial coefficient of $n$ and $k$, written as $\binom{n}{k}$ or $nC_k$ and often read as “$n$ choose $k$” (because it is the number of different ways of choosing $k$ different values from a set of $n$ different values, without repetition, ignoring order). It can also be defined or computed directly in terms of factorials:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial coefficients are related in many surprising ways — surprising, at least, until one works out the underlying algebra and recognizes that the relationships often just represent different ways of grouping factors together. One basic law hints at a recursive definition of binomial coefficients: For any natural numbers $n$ and $k$,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

One way to prove this is purely algebraic:

$$\binom{n+1}{k+1} = \frac{(n+1) \cdot n \cdot \ldots \cdot (n-k+1)}{(k+1) \cdot k \cdot \ldots \cdot 1}$$

$$= (n+1) \cdot \frac{n \cdot \ldots \cdot (n-k+1)}{(k+1) \cdot k \cdot \ldots \cdot 1}$$

$$= \frac{(k+1) + (n-k)}{(k+1) \cdot k \cdot \ldots \cdot 1} \cdot \frac{n \cdot \ldots \cdot (n-k+1)}{(k+1) \cdot k \cdot \ldots \cdot 1}$$

$$= \frac{n \cdot \ldots \cdot (n-k+1)}{(k+1) \cdot k \cdot \ldots \cdot 1} + \frac{n \cdot \ldots \cdot (n-k+1) \cdot (n-k)}{(k+1) \cdot k \cdot \ldots \cdot 1}$$

$$= \binom{n}{k} + \binom{n}{k+1}.$$

Another way to demonstrate the same result is combinatorial. Suppose that we have an organization $S$ with $n+1$ members, and we want to construct all of the possible committees that have $k+1$ members. Let’s designate one particular member $s$ of $S$ as special — perhaps she’s the senior member of the organization, for example. Now, in general, some of the $(k+1)$-member committees that we could construct from the members of $S$ will have this special individual $s$ as a member, and some will not.

First, then, how many will have the special value? We can answer this question by figuring out how many $k$-member committees we could construct from the $n$ other members of the organization. Having constructed any of these committees, we could extend it by appointing $s$ as well, resulting in a $(k+1)$-member committee with $s$ as a member. Since there are $\binom{n}{k}$ ways of choosing a $k$-member subcommittee from the other $n$ other members of the organization, there will also be $\binom{n}{k}$ ways of choosing a $(k+1)$-member committee that includes $s$ from the full membership of $S$.

How many $(k+1)$-member committees could one construct that would not have $s$ as a member? Well, each of these will be a $(k+1)$-member committee drawn from the $n$ other members of $S$, so there will be $\binom{n}{k+1}$ of them.
Thus the possible \((k + 1)\)-member committees of \(S\) comprise exactly \(\binom{n}{k}\) with \(s\) as a member, and exactly \(\binom{n}{k+1}\) without \(s\) as a member. By the Addition Rule, therefore, there will be \(\binom{n}{k} + \binom{n}{k+1}\) altogether. Thus \(\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}\). ■